

ON OPERATOR-VALUED SEMICIRCULAR RANDOM VARIABLES

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ABSTRACT. In this paper, we discuss some special properties of operator-valued semicircular random variables including representation of the Cauchy transform of a compactly supported probability measure in terms of their operator-valued Cauchy transforms and existence of nonzero discrete part of their associated distributions.

Keywords Semicircular distributions, Operator-Valued Non-Commutative Probability, Cauchy-Stieltjes transform, Continued Fractions

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1. INTRODUCTION

In his 1995 paper on operator-valued free probability theory, which became one of the main pillars of free probability, Voiculescu [11] introduced the operator-valued free central limit theorem and operator-valued semicircular random variables as operator-valued free analogues of the classical central limit theorem and normal random variables, respectively. Similar to normal random variables in classical probability theory, operator-valued semicircular random variables play key roles in many areas of operator-valued free probability and related fields. Following Wigner [12] and Voiculescu [10], Shlyakhtenko showed in [7] that operator-valued semicircular random variables also play an important role in random matrix theory, as asymptotic distributional limits of Gaussian band matrices. Later this result found several applications. We shall mention here the paper of Rashidi Far and others [6] which shows the importance of matrix-valued semicircular random variables in communication theory.

This paper deals with exploring some other aspects of operator-valued semicircular random variables. Our investigation originates from a question of Speicher on the existence of a discrete part in the spectrum of a matrix-valued semicircular random variable. Following well-established methods in noncommutative probability, we approach this problem through the study of distributions of such a random variable with respect to convenient positive linear functionals. In our work we were led to question roughly which probability measures can occur as scalar-valued distributions of operator-valued semicircular random variables. It turns out that

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the operator-valued semicircular random variables have a certain universal property: For given arbitrary compactly supported probability measure, its associated Cauchy transform is the composition of an extremal state and the operator-valued Cauchy transform of an infinite dimensional matrix-valued semicircular random variable (Theorem 3.1). Moreover, our proof gives a constructive method to find this semicircular random variable based on the continued fraction representation of the Cauchy transform of the given probability measure. Our second result directly concerns the discrete part of the distribution of a matrix-valued semicircular random variable with respect to the linear functional induced by the composition of the expectation with the matrix trace. It was noted by Speicher (private communications with Belinschi) that an $M_2(\mathbb{C})$ -valued semicircular random variable with variance $\eta \begin{pmatrix} z & v \\ y & w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z & v \\ y & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has a purely discrete distribution equal to $\mu = \frac{1}{4}\delta_{+1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{-1}$. Motivated by this observation and the proof of the Proposition 3.1, we were led to show in Theorem 4.1 that any matrix-valued semicircular random variable with nilpotent variance has an atom at zero. Finally, by combining the methods used in the proofs of the previous results, it is shown that the semicircular distributions of finite dimensional matrix-valued semicircular random variables can cover finitely supported probability measures in a special sense. This paper is divided into three sections including preliminaries, representation of the Cauchy transform using semicircular random variables and a discussion of atoms of distributions of matrix-valued semicircular random variables.

2. PRELIMINARIES

The reader who has studied the concepts of Cauchy or Cauchy-Stieltjes transform, continued fraction, operator-valued noncommutative random variables, and semicircular distributions is well acquainted with the following definitions and results. For an essential account of the mentioned concepts, see [4], [8], and [11]. Throughout this paper it is assumed that the probability measure μ on \mathbb{R} is compactly supported. We begin with a definition:

Definition 2.1. Let μ be a probability measure on the Borel σ -algebra of \mathbb{R} . The associated Cauchy transform G_μ to μ is defined by:

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \quad \text{Im}\xi \neq 0.$$

Some properties of G_μ are listed in the following proposition. We denote by $\mathbb{C}^+ = \{\xi \in \mathbb{C} : \text{Im}\xi > 0\}$, $\text{Supp}(\mu) = \mathbb{R} \setminus \bigcup\{U \subseteq \mathbb{R} | U \text{ open}, \mu(U) = 0\}$, and $\Gamma_\alpha(r) = \{\xi \in \mathbb{C}^+ : |\text{Re}(\xi) - r| < \alpha \text{Im}(\xi)\}$ ($-\infty < r < \infty$).

Proposition 2.1. Let $G = G_\mu$ be the Cauchy transform of a probability measure μ on \mathbb{R} . Then:

- (i) G is analytic on $\mathbb{C} \setminus \text{Supp}(\mu)$,
- (ii) If $\xi \in \mathbb{C}^+(\mathbb{C}^-)$, then $G(\xi) \in \mathbb{C}^-(\mathbb{C}^+)$,
- (iii) $\lim_{\Gamma_\alpha(0) \ni \xi \rightarrow \infty} \xi \cdot G(\xi) = 1$, for any fixed $0 < \alpha < \infty$.

Note that any function can be identified as a Cauchy transform of a probability measure on \mathbb{R} by possessing the three mentioned properties in the Proposition

above [8]. In addition, it is straightforward to check that:

$$(1) \quad \lim_{\Gamma_\alpha(r) \ni \xi \rightarrow r} (\xi - r)G_\mu(\xi) = \mu(\{r\}) \quad (-\infty < r < \infty).$$

Definition 2.2. A probability measure μ on \mathbb{R} is said to have finite moment of order $m \geq 1$ if $\int_{\mathbb{R}} |t|^m d\mu(t) < \infty$, and in this case the m^{th} moment of μ is defined by $M_m = \int_{\mathbb{R}} t^m d\mu(t)$.

We denote the set of all Borel probability measures on \mathbb{R} having finite moments of all orders by $\mathcal{B}_{fm}(\mathbb{R})$. It is trivial that any compactly supported probability measure μ is in this set, and as a corollary of the Carleman's moment test [4, Theorem 1.36] it is a solution of the determinate moment problem. Next, using the idea of associated Gram-Schmidt orthonormal polynomials to a probability measure μ , one can prove the existence of the so-called Jacobi coefficients of the given probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$, [4, Theorem 1.44]:

Theorem 2.3. Let $\{p_m(t)\}_{m=0}^\infty$ be the Gram-Schmidt orthonormal polynomials associated with given $\mu \in \mathcal{B}_{fm}(\mathbb{R})$. Then there exists a pair of sequences $\{\alpha_m\}_{m=1}^\infty \subseteq \mathbb{R}$ and $\{\omega_m\}_{m=1}^\infty \subseteq \mathbb{R}^+$ uniquely determined by:

$$(2) \quad \begin{aligned} p_0(t) &= 1, \\ p_1(t) &= t - \alpha_1, \\ tp_m(t) &= p_{m+1}(t) + \alpha_{m+1}p_m(t) + \omega_m p_{m-1}(t) \quad m \geq 1, \end{aligned}$$

where in which, if $|Supp(\mu)| = \infty$, both $\{\alpha_m\}_{m=1}^\infty, \{\omega_m\}_{m=1}^\infty$ are infinite sequences, and if $|Supp(\mu)| = m_0 < \infty$, we have $\{\alpha_m\}_{m=1}^\infty = \{\alpha_m\}_{m=1}^{m_0}$ and $\{\omega_m\}_{m=1}^\infty = \{\omega_m\}_{m=1}^{m_0-1}$ with $p_{m_0} = 0$.

Note that as a corollary of the equations (2), for the compactly supported probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ we have:

$$(3) \quad \sup_m (|\alpha_m| + \omega_m) \leq 2 \sup_{t \in Supp(\mu)} |t| < \infty.$$

The following theorem gives a continued fraction representation of the associated Cauchy transform of any probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$, [4, Theorem 1.97]:

Theorem 2.4. Let $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ and $(\{\omega_n\}_{n=1}^\infty, \{\alpha_n\}_{n=1}^\infty)$ be its Jacobi coefficients. If μ is the solution of the determinate moment problem, then the Cauchy transform of it is expanded into a continued fraction

$$G_\mu(\xi) = \frac{1}{\xi - \alpha_1 - \frac{\omega_1}{\xi - \alpha_2 - \frac{\omega_2}{\xi - \alpha_3 - \ddots}}} \quad Im(\xi) \neq 0.$$

Before introducing the distribution of operator-valued semicircular random variables, we remind the following essential definitions and results from the operator-valued noncommutative probability theory:

Definition 2.5. (1) Let A be a unital $*$ -algebra and let B denote a fixed unital $*$ -subalgebra of A over \mathbb{C} . A linear map $E_B : A \rightarrow B$ is called a conditional expectation if it satisfies the following conditions:

- (i) $E_B(b_1 a b_2) = b_1 E_B(a) b_2$ for all $a \in A$, $b_1, b_2 \in B$, and $E_B(1) = 1$,
(ii) $E_B(a^* a) \geq 0$, for all $a \in A$,

(2) A triple (A, E_B, B) as in part (1) is called a B -valued non-commutative probability space. An element $a \in A$ is called a B -valued random variable,
(3) Let (A, E_B, B) be as in part (2), and $B \subseteq A_i \subseteq A (i \in I)$ be subalgebras. The family $\{A_i\}_{i \in I}$ is called free over B if

$$E_B(a_{i_1} a_{i_2} \cdots a_{i_n}) = 0$$

whenever $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n, a_{i_j} \in A_{i_j}$ and $E_B(a_{i_j}) = 0, (1 \leq j \leq n)$.

We call the family $\{X_i\}_{i \in I}$ of subsets of A (elements $\{a_i\}_{i \in I}$ of A) free if the corresponding family of subalgebras $\{\langle X_i \cup B \rangle\}_{i \in I}$ ($\{\langle a_i \rangle \cup B\}_{i \in I}$) is free.

Given an operator-valued noncommutative probability space (A, E_B, B) and a B -valued random variable $a \in A$, the associated moments of a are, by definition, the multilinear functionals $\{m_n\}_{n=0}^\infty$ defined by:

$$\begin{aligned} m_n : B^n &\rightarrow B, \\ m_n(b_1, b_2, \dots, b_n) &= E_B(ab_1 ab_2 \cdots ab_n a), \end{aligned}$$

in which the quantity $m_0 = E_B(a) \in B$ is called the first moment, the map $b_1 \mapsto E_B(ab_1 a)$ is called the second moment, and in general the map $(b_1, b_2, \dots, b_n) \mapsto E_B(ab_1 ab_2 \cdots ab_n a)$ is called the $(n+1)^{th}$ moment. Next, let B be a Banach algebra, and $a, a_1, \dots, a_m, \dots$ be a sequence of B -valued random variables in A with associated sequences of moments $\{m_n\}_{n=0}^\infty, \{m_n^{(1)}\}_{n=0}^\infty, \dots, \{m_n^{(m)}\}_{n=0}^\infty, \dots$, respectively. We say that the sequence $\{a_m\}_{m=1}^\infty$ convergence to a in moments if

$$\lim_{m \rightarrow \infty} \|m_n^{(m)}(b_1, b_2, \dots, b_n) - m_n(b_1, b_2, \dots, b_n)\| = 0,$$

for all $(b_1, b_2, \dots, b_n) \in B^n$, and $n \geq 0$.

Having the same assumptions as above, we recall that any $a \in A$ can be written as $a = Re(a) + i.Im(a)$ where $Re(a) = \frac{a+a^*}{2}$ and $Im(a) = \frac{a-a^*}{2i}$ are self-adjoint elements. We define $\mathbb{H}^+(A) = \{a \in A | Im(a) > 0\}$, where $Im(a) > 0$ means $Im(a) > \epsilon.1$ for some $\epsilon > 0$, and similarly $\mathbb{H}^+(B)$. Then the operator-valued Cauchy transform of $a \in A$ is an analytic map G_a defined via:

$$\begin{aligned} G_a : \mathbb{H}^+(B) &\rightarrow \mathbb{H}^-(B) \\ G_a(b) &= E_B((b-a)^{-1}) \\ &= \sum_{n=0}^{\infty} b^{-1} E_B((ab^{-1})^n) \quad b \in \mathbb{H}^+(B), \|b^{-1}\| < \|a\|^{-1}. \end{aligned}$$

Next, the operator-valued R -transform of $a \in A$, $R_a : B \rightarrow B$, can be defined by the relation

$$bG_a(b) = 1 + R_a(G_a(b)).G_a(b) \quad (b \in B),$$

[11, Theorem 4.9.]. The following central limit theorem for operator-valued random variables is due to Voiculescu, [11, Theorem 8.4.]:

Theorem 2.6. (*Free Central Limit Theorem*) Let B be a Banach algebra and $a_1, a_2, \dots, a_m, \dots$ be a sequence of free B -valued random variables in the non-commutative operator valued probability space (A, E_B, B) such that:

(i) $E_B(a_m) = 0$, ($m \in \mathbb{N}$),

(ii) there is a bounded linear map $\eta : B \rightarrow B$ such that

$$\lim_{n \rightarrow \infty} \frac{\sum_{m=1}^n E_B(a_m b a_m)}{n} = \eta(b), \quad (b \in B),$$

(iii) there are constants C_k ($k \in \mathbb{N}$) such that

$$\sup_{m \in \mathbb{N}} \|E_B(a_m b_1 a_m \cdots b_k a_m)\| \leq C_k \|b_1\| \cdots \|b_k\| \quad (k \in \mathbb{N}).$$

Then the sequence $S_m = \frac{\sum_{k=1}^m a_k}{\sqrt{m}}$ ($m \in \mathbb{N}$) converges in moments.

The central limit in the above theorem which we shall denote it by s is called a B -valued semicircular element in the context of operator-valued noncommutative probability, and, as in classical probability theory it is uniquely determined by its first two moments. Indeed, its R -transform is of the form

$$R_s(b) = D + \eta(b), \quad (b \in B)$$

where $D = E_B(s) \in B$ is a self-adjoint element and $\eta : B \rightarrow B$ is a completely positive map given by $\eta(b) = E_B(sbs) - E_B(s)bE_B(s)$ ($b \in B$). Furthermore, for any completely positive map $\eta : B \rightarrow B$ there exists a B -valued semicircular random variable s such that $\eta(b) = E_B(sbs) - E_B(s)bE_B(s)$ ($b \in B$), [9, Theorem 4.3.1.].

The following result of Helton, Rashidi Far, and Speicher, shows that any operator-valued semicircular random variable can be uniquely determined by a functional equation involving only its first two moments, [3]:

Theorem 2.7. *Let A be a unital C^* -algebra, B a C^* -subalgebra of A , and $s \in A$ be a self-adjoint B -valued semicircular random variable with first moment $D \in B$ and variance $\eta : B \rightarrow B$. Then its associated operator-valued Cauchy transform $G_s : \mathbb{H}^+(B) \rightarrow \mathbb{H}^-(B)$ is the unique solution of the functional equation*

$$(4) \quad b.G_s(b) = 1 + (D + \eta(G_s(b))).G_s(b), \quad (b \in \mathbb{H}^+(B)),$$

together with asymptotic condition

$$(5) \quad \lim_{b^{-1} \rightarrow 0} b.G_s(b) = 1.$$

Let A be a unital C^* -algebra, B be a unital C^* -subalgebra of A , $E_B : A \rightarrow B$ be a conditional expectation, and $G_a : \mathbb{H}^+(B) \rightarrow \mathbb{H}^-(B)$ be the operator-valued Cauchy transform of the self-adjoint random variable $a \in A$. Let $\Phi : B \rightarrow \mathbb{C}$ be a given state on the C^* -algebra B (In the case of $B = M_n(\mathbb{C})$, for some fixed $n \geq 1$, we can take $\Phi = tr_n$, the normalized trace.), and define a map $G : \mathbb{C}^+ \rightarrow \mathbb{C}^-$ via:

$$G(\xi) = (\Phi \circ G_a)(\xi.1) \quad \xi \in \mathbb{C}^+.$$

Referring to the note after the Proposition 2.1, it follows that there is a probability measure $\mu = \mu_a \in \mathcal{B}_{fm}(\mathbb{R})$ on \mathbb{R} , which we call the distribution of a , such that:

$$(6) \quad G(\xi) = G_{\mu_a}(\xi) = \int_{\mathbb{R}} \frac{d\mu_a(t)}{\xi - t} \quad \xi \in \mathbb{C}^+.$$

Note that the atoms of μ_a are determined via Equation (1).

3. REPRESENTATION OF THE CAUCHY TRANSFORM USING SEMICIRCULAR RANDOM VARIABLES

The proof of the following results are based on the Theorem 2.7 of Section 2. Indeed, for given D and η as in that theorem, we shall verify the conditions (4) and (5) for $b = \xi.1$ $\xi \in \mathbb{C}^+$ and $G_s(b)$ a diagonal matrix of complex analytic functions. Then, Theorem 2.7 of Section 2 will guarantee us that there is a semicircular random variable s with first moment D and variance η so that $G_s(b)$ is the restriction of the operator-valued Cauchy transform of s to \mathbb{C}^+ . Next, our variances η will be explicitly constructed as $\eta(b) = v^*bv$, [5, Theorem 4.1.] with v obtained from the Jacobi coefficients of the given compactly supported probability measure.

The first result deals with the finite dimensional matrix-valued representations of the Cauchy transform. Here, we consider ℓ_2^n as \mathbb{C}^n with its canonical orthonormal basis $\{e_k\}_{k=1}^n$.

Proposition 3.1. *Let μ be a probability measure with compact support in \mathbb{R} . Then there exist two sequences $s_n^{(1)}$ and $s_n^{(2)}$ ($n \geq 1$) of self-adjoint operator valued semicircular random variables with associated operator-valued Cauchy transforms $G_{s_n^{(1)}} : \mathbb{H}^+(M_n(\mathbb{C})) \rightarrow \mathbb{H}^-(M_n(\mathbb{C}))$ and $G_{s_n^{(2)}} : \mathbb{H}^+(M_n(\mathbb{C})) \rightarrow \mathbb{H}^-(M_n(\mathbb{C}))$ ($n \geq 1$) such that the Cauchy transform $G_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^-$ is represented as:*

$$(7) \quad G_\mu(\xi) = \lim_{n \rightarrow \infty} \langle G_{s_n^{(1)}}(\xi.1_n)e_1, e_1 \rangle_{\ell_2^n} \quad \xi \in \mathbb{C}^+,$$

and

$$(8) \quad G_\mu(\xi) = \lim_{n \rightarrow \infty} \langle G_{s_n^{(2)}}(\xi.1_n)e_n, e_n \rangle_{\ell_2^n} \quad \xi \in \mathbb{C}^+.$$

Proof. Let

$$G_\mu(\xi) = \frac{1}{\xi - \alpha_1 - \frac{\omega_1}{\xi - \alpha_2 - \frac{\omega_2}{\xi - \alpha_3 - \dots - \frac{\omega_{n-1}}{\xi - \alpha_n - \frac{\omega_n}{\xi - \alpha_{n+1} - \dots}}}}}$$

be the continued fraction representation of G_μ as in Theorem 2.4. To prove Equation (7), fix positive integer $n \geq 1$, then define $b = \xi.1_n$, $D_n^{(1)} = (\alpha_k \delta_{kl})_{k,l=1}^n$ and the completely positive map $\eta_n^{(1)}$ via :

$$\begin{aligned} \eta_n^{(1)} : M_n(\mathbb{C}) &\rightarrow M_n(\mathbb{C}) \\ \eta_n^{(1)}((a_{kl})_{k,l=1}^n) &= (\omega_k^{\frac{1}{2}} \delta_{(k+1)l})_{k,l=1}^n (a_{kl})_{k,l=1}^n (\omega_{k-1}^{\frac{1}{2}} \delta_{k(l+1)})_{k,l=1}^n. \end{aligned}$$

Then, for the self-adjoint semicircular element $s_n^{(1)}$ with operator-valued Cauchy transform $G_{s_n^{(1)}}$ satisfying the functional equation (4) of the form:

$$bG_{s_n^{(1)}}(b) = 1 + (D_n^{(1)} + \eta_n^{(1)}(G_{s_n^{(1)}}(b)))G_{s_n^{(1)}}(b),$$

we have $G_{s_n^{(1)}}(b) = (g_{n,n-k+1}(\xi)\delta_{kl})_{k,l=1}^n$ where :

$$g_{n,n-k+1}(\xi) = \frac{1}{\xi - \alpha_1 - \frac{\omega_1}{\xi - \alpha_2 - \frac{\omega_2}{\xi - \alpha_3 - \dots - \frac{\omega_{n-k+1}}{\xi - \alpha_{n-k+1}}}}} \quad 1 \leq k \leq n, \xi \in \mathbb{C}^+,$$

which can be identified as $(n-k+1)^{th}$ convergent of G_μ . Accordingly:

$$G_\mu(\xi) = \lim_{n \rightarrow \infty} g_{n,n}(\xi) = \lim_{n \rightarrow \infty} \langle G_{s_n^{(1)}}(\xi.1_n)e_1, e_1 \rangle_{\ell_2^n} \quad \xi \in \mathbb{C}^+.$$

The proof of Equation (8) is analogous by considering a fixed positive integer $n \geq 1$, then defining $b = \xi.1_n$, $D_n^{(2)} = (\alpha_{n+1-k}\delta_{kl})_{k,l=1}^n$ and the completely positive map $\eta_n^{(2)}$ via :

$$\begin{aligned} \eta_n^{(2)} : M_n(\mathbb{C}) &\rightarrow M_n(\mathbb{C}) \\ \eta_n^{(2)}((a_{kl})_{k,l=1}^n) &= (\omega_{n-k+1}^{\frac{1}{2}}\delta_{k(l+1)})_{k,l=1}^n (a_{kl})_{k,l=1}^n (\omega_{n-k}^{\frac{1}{2}}\delta_{(k+1)l})_{k,l=1}^n. \end{aligned}$$

□

The following theorem deals with the infinite dimensional matrix valued representation of the Cauchy transform. Here, we denote by $B(\ell_2(\mathbb{N}))$ the space of bounded operators on the separable Hilbert space $\ell_2(\mathbb{N})$, and we consider the orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\ell_2(\mathbb{N})$ given by $e_n = \{\delta_{mn}\}_{m=1}^\infty$ ($n \geq 1$).

Theorem 3.1. *Let μ be a compactly supported probability measure in \mathbb{R} with Jacobi coefficients $(\{\omega_n\}_{n=1}^\infty, \{\alpha_n\}_{n=1}^\infty)$. Then there exist a self-adjoint $B(\ell_2(\mathbb{N}))$ -valued semicircular random variable s with first moment $D = (\alpha_k\delta_{kl})_{k,l=1}^\infty \in B(\ell_2(\mathbb{N}))$, and variance*

$$\begin{aligned} \eta : B(\ell_2(\mathbb{N})) &\rightarrow B(\ell_2(\mathbb{N})) \\ \eta((a_{kl})_{k,l=1}^\infty) &= (\omega_k^{\frac{1}{2}}\delta_{k(l+1)})_{k,l=1}^\infty (a_{kl})_{k,l=1}^\infty (\omega_{k-1}^{\frac{1}{2}}\delta_{k(l+1)})_{k,l=1}^\infty, \end{aligned}$$

and an state $\rho : B(\ell_2(\mathbb{N})) \rightarrow \mathbb{C}$ such that the Cauchy transform $G_\mu : \mathbb{C}^+ \rightarrow \mathbb{C}^-$ is represented as:

$$G_\mu(\xi) = (\rho \circ G_s)(\xi.1) \quad \xi \in \mathbb{C}^+.$$

Proof. Let

$$G_\mu(\xi) = \frac{1}{\xi - \alpha_1 - \frac{\omega_1}{\xi - \alpha_2 - \frac{\omega_2}{\xi - \alpha_3 - \dots - \frac{\omega_{n-1}}{\xi - \alpha_n - \frac{\omega_n}{\xi - \alpha_{n+1} - \dots}}}}}$$

be the continued fraction representation of G_μ as in Theorem 2.4 and consider given D and η where using inequality (3) both of them are bounded in their corresponding norms. Now, as in proof of Proposition 3.1, we observe that the diagonal matrix

$$G_s(b) = (x_{kl}\delta_{kl})_{k,l=1}^\infty, \quad b = \xi.1 \in B(\ell_2(\mathbb{N}))$$

with entries x_{nn} of the form

$$x_{nn} = \frac{1}{\xi - \alpha_n - \frac{\omega_n}{\xi - \alpha_{n+1} - \frac{\omega_{n+1}}{\xi - \alpha_{n+2} - \frac{\omega_{n+2}}{\xi - \alpha_{n+3} - \frac{\omega_{n+3}}{\xi - \alpha_{n+4} - \dots}}}}} \quad n \geq 1,$$

satisfies the equation (4) with condition (5). Consequently, for the state $\rho : B(\ell_2(\mathbb{N})) \rightarrow \mathbb{C}$ defined by:

$$\rho(T) = \langle T(e_1), e_1 \rangle_{\ell_2(\mathbb{N})}$$

the assertion follows. \square

4. ATOMS OF DISTRIBUTIONS OF MATRIX-VALUED SEMICIRCULAR RANDOM VARIABLES

In this section, the existence of atoms of distributions of finite dimensional matrix-valued semicircular random variables is discussed. First of all, we give a sufficient condition on the variance of a centered semicircular random variable so that its associated probability measure has atom.

Theorem 4.1. *Let $G_s : \mathbb{H}^+(M_n(\mathbb{C})) \rightarrow \mathbb{H}^-(M_n(\mathbb{C}))$ be the operator valued Cauchy transform of a $M_n(\mathbb{C})$ -valued semicircular random variable s satisfying the functional equation (4), $b \in \mathbb{H}^+(M_n(\mathbb{C}))$, where $D = 0$ and $\eta : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a nilpotent completely positive map. Then the associated probability measure μ_s to G_s has at least one atom.*

Proof. Let $\eta, \dots, \eta^{m-1} \neq 0$ and $\eta^m = 0$, for some $m \geq 1$. Writing the functional equation in the form of $b - G_s(b)^{-1} = \eta(G_s(b))$ for $\|b^{-1}\| < \infty$ it follows that:

$$\eta^{m-1}(b - G_s(b)^{-1}) = \eta^m(G_s(b)) = 0 \quad \text{Im}(b) > 0, \quad \|b^{-1}\| < \infty.$$

Now, if $\ker(\eta^{m-1}) = 0$, then $G_s(b) = b^{-1}$ and using Equations (1) and (6) it follows that

$$\mu_s(\{0\}) = 1,$$

proving the assertion. Hence, we may assume $\ker(\eta^{m-1}) \neq 0$. Pick $0 \neq c \in \ker(\eta^{m-1}) \cap M_n^+(\mathbb{C})$ with $\|c\| = 1$. Then by Schwarz inequality for completely positive maps [5, p. 40], it follows that:

$$\eta^{m-1}(c^{\frac{1}{2^n}})^* \eta^{m-1}(c^{\frac{1}{2^n}}) \leq \|\eta^{m-1}(1)\| \eta^{m-1}(c^{\frac{1}{2^{n-1}}}) \quad n \geq 1,$$

and by induction we conclude that $\eta^{m-1}(c^{\frac{1}{2^n}}) = 0$ ($n \geq 1$). By defining:

$$p := \text{s.o.t.} \lim_{n \rightarrow \infty} c^{\frac{1}{2^n}},$$

it follows that p is a projection in $\ker(\eta^{m-1})$.

Claim (1): There exists a unique projection $1 \neq q \in \ker(\eta^{m-1})$ such that for any projection $p \in \ker(\eta^{m-1})$ we have: $p \leq q$.

We showed that there is at least one projection p in $\ker(\eta^{m-1})$. Let p_1, p_2 be two projections in $\ker(\eta^{m-1})$. Then :

$$\eta^{m-1}((p_1 + p_2)^{\frac{1}{2^n}})^* \eta^{m-1}((p_1 + p_2)^{\frac{1}{2^n}}) \leq \|\eta^{m-1}(1)\| \eta^{m-1}((p_1 + p_2)^{\frac{1}{2^{n-1}}}) \quad n \geq 1,$$

and by induction it follows that $\eta^{m-1}((p_1 + p_2)^{\frac{1}{2^n}}) = 0$ ($n \geq 1$). Now, define $p_3 := \text{s.o.t.} \lim_{n \rightarrow \infty} (p_1 + p_2)^{\frac{1}{2^n}}$, then p_3 is a projection in $\ker(\eta^{m-1})$. On the other hand

$$(p_1 + p_2)^{\frac{1}{2^n}} \geq p_1, p_2 \quad (n \geq 1),$$

yielding $p_3 \geq p_1, p_2$. Next, using the maximality argument and by repeating this process there will be a unique maximal projection q in $\ker(\eta^{m-1})$ such that for any other projection p in it, we have $p \leq q$. Finally, if $q = 1$, then using the same Schwarz inequality as above, and the canonical decomposition of elements of $M_n(\mathbb{C})$ into its positive elements it follows that $\eta^{m-1} = 0$, in contradiction to our hypothesis.

Claim(2): For any $0 \neq c \in \ker(\eta^{m-1}) \cap M_n^+(\mathbb{C})$, we have $cq = qc = c$.

Indeed, since $c \in M_n^+(\mathbb{C})$, by spectral theorem we have:

$$c = \sum_{k=1}^N \lambda_k p_k$$

where the projections p_k 's satisfy $\sum_{k=1}^N p_k = 1$, $p_{k_1} p_{k_2} = 0$ ($1 \leq k_1 \neq k_2 \leq N$) and $\lambda_k \geq 0$ ($1 \leq k \leq N$). Now, define:

$$r := \text{s.o.t.} \lim_{n \rightarrow \infty} c^{\frac{1}{2^n}}.$$

Then it follows that $r = \sum_{\lambda_k \neq 0} p_k$, yielding $p_k \leq r$ ($\lambda_k \neq 0$). On the other hand, by definition of q we have $r \leq q$ and hence $p_k \leq q$ ($\lambda_k \neq 0$). But all of $p_k \leq q$ ($\lambda_k \neq 0$), and q are projections and, consequently, $p_k q = q p_k = p_k$ ($\lambda_k \neq 0$). Now, by multiplying all sides by λ_k ($1 \leq k \leq N$), and taking summation the claim is proved.

Next, take $c_0 = \frac{1}{i}(G_s(b)^{-1} - b)$ where $b = iy.1$ ($y > 0$). Then using the fact that s is centered, $E_B(s^{2m-1}) = 0$ $m \geq 1$, by

$$\begin{aligned} \operatorname{Re}(G_s(b)) &= \operatorname{Re}\left(\sum_{m=0}^{\infty} b^{-1} E_B((sb^{-1})^m)\right) = \operatorname{Re}\left(\sum_{m=0}^{\infty} i^{m+1} (-y)^{m+1} E_B(s^m)\right) \\ &= \sum_{m=1}^{\infty} (-y^2)^m E_B(s^{2m-1}) = 0, \quad y > \|s\| \end{aligned}$$

it follows that

$$G_s(b)^{-1} - b = \left(\operatorname{Re}(G_s(b)) + i \operatorname{Im}(G_s(b))\right)^{-1} - b = i \cdot \left(-\operatorname{Im}(G_s(b))^{-1} - \frac{b}{i}\right),$$

and, hence $c_0 = \frac{1}{i}(G_s(b)^{-1} - b) = \operatorname{Im}((G_s(b)^{-1} - b)) \geq 0$. Now, by claim (2) for $c = c_0$ we have that:

$$G_s(b)(1 - q) = b^{-1}(1 - q) = (1 - q)G_s(b) \quad \text{and} \quad G_s(b)q = qG_s(b),$$

and hence:

$$\begin{aligned} G_s(b) &= (1-q)G_s(b)(1-q) + qG_s(b)q \\ &= b^{-1}(1-q) + qG_s(b)q \quad b = iy.1 \ (y > 0), \|b^{-1}\| < \infty. \end{aligned}$$

Now, applying analytic continuation for the complex function $tr_n \circ G_s|_{\mathbb{C}^+} : \mathbb{C}^+ \rightarrow \mathbb{C}^-$ we conclude that:

$$(tr_n \circ G_s)(\xi.1) = tr_n(\xi^{-1}(1-q) + qG_s(\xi.1)q) \quad \xi \in \mathbb{C}^+,$$

and, consequently, by another application of the Equations (1) and (6):

$$\mu_s(\{0\}) = tr_n(1-q) > 0,$$

completing the proof. \square

Before stating a more concrete and special case of the above theorem, using M. D. Choi's representation of a completely positive map from a matrix algebra to another matrix algebra [1], we have:

Lemma 4.2. *Let η be a completely positive map defined by*

$$\begin{aligned} \eta : M_n(\mathbb{C}) &\rightarrow M_n(\mathbb{C}) \\ \eta(a) &= \sum_{j=1}^{n^2} a_j a a_j^*, \quad a_j \in M_n(\mathbb{C}) \ (1 \leq j \leq n^2). \end{aligned}$$

Then:

- (i) *if the map η is nilpotent, then all matrices a_j ($1 \leq j \leq n^2$) are nilpotent,*
- (ii) *if all matrices a_j ($1 \leq j \leq n^2$) are nilpotent and commute with each other, then the map η is nilpotent.*

Proof. (i) Let $\eta^m = 0$ for some $m \geq 1$. Then, it follows that:

$$\begin{aligned} \sum_{1 \leq j_1, \dots, j_m \leq n^2} (a_{j_1} \cdots a_{j_m} a)(a_{j_1} \cdots a_{j_m} a)^* &= \sum_{1 \leq j_1, \dots, j_m \leq n^2} (a_{j_1} \cdots a_{j_m}) a a^* (a_{j_m}^* \cdots a_{j_1}^*) \\ &= \sum_{j_1=1}^{n^2} \cdots \sum_{j_m=1}^{n^2} (a_{j_1} \cdots a_{j_m}) a a^* (a_{j_m}^* \cdots a_{j_1}^*) \\ &= \sum_{j_1=1}^{n^2} a_{j_1} \left(\cdots \left(\sum_{j_m=1}^{n^2} a_{j_m} a a^* a_{j_m}^* \right) \cdots \right) a_{j_1}^* \\ &= \eta^m(a a^*) \\ &= 0 \end{aligned}$$

for all $a \in M_n(\mathbb{C})$. Consequently, by positivity of all elements of the form $(a_{j_1} \cdots a_{j_m} a)(a_{j_1} \cdots a_{j_m} a)^*$ it follows that

$$(a_{j_1} \cdots a_{j_m} a)(a_{j_1} \cdots a_{j_m} a)^* = 0 \quad (1 \leq j_1, \dots, j_m \leq n^2),$$

for all $a \in M_n(\mathbb{C})$. On the other hand, $M_n(\mathbb{C})$ is a C^* -algebra and hence:

$$a_{j_1} \cdots a_{j_m} a = 0 \quad (1 \leq j_1, \dots, j_m \leq n^2),$$

for all $a \in M_n(\mathbb{C})$, or equivalently:

$$a_{j_1} \cdots a_{j_m} = 0 \quad (1 \leq j_1, \dots, j_m \leq n^2).$$

Now, take $j_1 = \dots = j_m = j$ where $1 \leq j \leq n^2$ and the desired result is proved.

(ii) Since a_j ($1 \leq j \leq n^2$) are nilpotent, it follows that $a_j^n = 0$ ($1 \leq j \leq n^2$). Put $m = n^3$, then by commutativity of these matrices it follows that:

$$a_{j_1} \cdots a_{j_m} = \prod_{p=0}^{n-1} \prod_{q=1}^{n^2} a_{j_{pn^2+q}} = a_{j_{(j_1, \dots, j_m)}}^n \prod_{j_{pn^2+q} \neq j_{(j_1, \dots, j_m)}} a_{j_{pn^2+q}} = 0,$$

for all $1 \leq j_1, \dots, j_m \leq n^2$. Consequently,

$$\eta^m(a) = \sum_{1 \leq j_1, \dots, j_m \leq n^2} (a_{j_1} \cdots a_{j_m}) a(a_{j_m}^* \cdots a_{j_1}^*) = 0$$

for all $a \in M_n(\mathbb{C})$. \square

Considering above lemma, for a category of nilpotent η 's we have:

Corollary 4.3. *Let $G_s : \mathbb{H}^+(M_n(\mathbb{C})) \rightarrow \mathbb{H}^-(M_n(\mathbb{C}))$ be the operator-valued Cauchy transform of a $M_n(\mathbb{C})$ -valued semicircular random variable s satisfying the functional equation (4), $b \in \mathbb{H}^+(M_n(\mathbb{C}))$, $D = 0$ and the completely positive map η is given by:*

$$\eta : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

$$\eta(a) = \sum_{j=1}^{n^2} a_j a a_j^*, \quad a_j^n = 0 \quad (1 \leq j \leq n^2), \quad a_{j_1} a_{j_2} = a_{j_2} a_{j_1} \quad (1 \leq j_1, j_2 \leq n^2).$$

Then the associated probability measure μ_s to G_s has at least one atom at $x = 0$.

A few examples of interest are discussed in connection to the Theorem 4.1. :

Remarks 4.4. (i) *The converse of the assertion of the Theorem 4.1. does not hold. To see this, let $n = 2$, $0 \neq |\alpha| \neq |\beta| \neq 0$ and define:*

$$a_1 = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \quad a_j = 0 \quad (2 \leq j \leq 4).$$

Under these conditions, considering $D = 0$ and $\eta(a) = a_1 a a_1^*$ in the Equation (4) with the restriction condition (5), and solving it for $G_s(b)$ yields:

$$tr_2(G_s(b)) = \frac{(|\alpha|^2 + |\beta|^2) \left(\xi^2 + \sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4|\alpha|^2 \xi^2} \right) - (|\alpha|^2 - |\beta|^2)^2}{4\xi|\alpha|^2|\beta|^2},$$

and, by considering the Equations (1) and (6) it follows that:

$$\mu_s(\{r\}) = 0 \quad \text{if } r \neq 0, \quad \frac{1}{2} \left(1 - \left| \frac{\beta}{\alpha} \right|^{2sgn(1 - |\frac{\beta}{\alpha}|)} \right) \quad \text{if } r = 0.$$

(ii) *If the assumption of nilpotency of the map η is violated in the statement of the Theorem 4.1, then the associated distribution can have no atom. To see this, let $n = 2$, $|\alpha| = |\beta| \neq 0$ and define:*

$$a_1 = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \quad a_j = 0 \quad (2 \leq j \leq 4).$$

Under these conditions, considering $D = 0$ and $\eta(a) = a_1 a a_1^*$ in the Equation (4) with the restriction condition (5), and solving it for $G_s(b)$ yields:

$$\text{tr}_2(G_s(b)) = \frac{\xi - \sqrt{\xi^2 - 4|\alpha|^2}}{2|\alpha|^2}$$

Now, by considering the Equation (6), it follows that this is the Cauchy transform of the standard semicircular law of Wigner which has no atom.

(iii) In the Theorem 4.1, for given nilpotent map η the associated probability measure to the operator-valued Cauchy transform $G_s(b)$ may not be purely atomic. To see this, let $n = 3$ and define :

$$a_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_j = 0 \quad (3 \leq j \leq 9).$$

Under these conditions, considering $D = 0$ and $\eta(a) = a_1 a a_1^*$ in the Equation (4) with the restriction condition (5), and solving it for $G_s(b)$ with the Groebner basis method [2] in Mathematica, it follows that:

$$\text{tr}_3(G_s(b)) = \frac{1}{3} \left(\frac{1}{\xi} + \frac{\xi - \sqrt{\xi^2 - 4}}{2} + (\xi^2 - 1) \left(\frac{\xi - \sqrt{\xi^2 - 4}}{2} \right)^3 \right),$$

and, by considering the Equations (1) and (6) it follows that:

$$\mu_s(\{r\}) = 0 \quad \text{if } r \neq 0, \quad \frac{1}{3} \quad \text{if } r = 0,$$

showing that μ_s is not purely atomic.

We end this section by mentioning a “covering property” of distributions of matrix-valued semicircular random variables. Before that, we need a definition:

Definition 4.5. Let μ and ν be two probability measures on \mathbb{R} . We shall say that ν is a component of μ if there exists a finite family $\{\nu_1, \dots, \nu_n\}$ of probability measures so that $\nu \in \{\nu_1, \dots, \nu_n\}$ and

$$\mu = \sum_{j=1}^n \alpha_j \nu_j$$

for some $\alpha_1, \dots, \alpha_n \in [0, 1]$ satisfying $\alpha_1 + \dots + \alpha_n = 1$.

Proposition 4.1. Any probability measure μ on \mathbb{R} whose support is a finite set can be realized as a component of a semicircular distribution μ_s of some matrix-valued semicircular random variable s with nilpotent variance.

Proof. Assume $|\text{Supp}(\mu)| = n < \infty$, and let

$$G_\mu(\xi) = \frac{1}{\xi - \alpha_1 - \frac{\omega_1}{\xi - \alpha_2 - \frac{\omega_2}{\xi - \alpha_3 - \dots - \frac{\omega_{n-1}}{\xi - \alpha_n}}}}$$

be the continued fraction representation of G_μ as in Theorem 2.4. Then as in the proof of the Proposition 3.1, define $b = \xi.1_n$, $D_n = (\alpha_k \delta_{kl})_{k,l=1}^n$ and the nilpotent completely positive map η_n via:

$$\begin{aligned} \eta_n : M_n(\mathbb{C}) &\rightarrow M_n(\mathbb{C}) \\ \eta_n((a_{kl})_{k,l=1}^n) &= (\omega_k^{\frac{1}{2}} \delta_{(k+1)l})_{k,l=1}^n (a_{kl})_{k,l=1}^n (\omega_{k-1}^{\frac{1}{2}} \delta_{k(l+1)})_{k,l=1}^n. \end{aligned}$$

Then for the self-adjoint semicircular element $s = s_n$ with operator-valued Cauchy transform G_s satisfying the functional equation (4) in the form:

$$bG_s(b) = 1 + (D_n + \eta_n(G_s(b)))G_s(b),$$

we have

$$G_\mu(\xi) = \langle G_s(\xi.1_n)e_1, e_1 \rangle_{\ell_2^n} \quad \xi \in \mathbb{C}^+.$$

Next, let μ_k ($1 \leq k \leq n$) be a finitely supported probability measure on \mathbb{R} with associated Cauchy transform:

$$G_{\mu_{n-(k-1)}}(\xi) = \frac{1}{\xi - \alpha_1 - \frac{\omega_1}{\xi - \alpha_2 - \frac{\omega_2}{\xi - \alpha_3 - \dots - \frac{\omega_{k-1}}{\xi - \alpha_k}}}} \quad (1 \leq k \leq n).$$

Note that $\mu = \mu_1$. Then by proof of the Proposition 3.1, we have

$$(9) \quad G_{\mu_k}(\xi) = \langle G_s(\xi.1_n)e_k, e_k \rangle_{\ell_2^n} \quad (1 \leq k \leq n), \quad \xi \in \mathbb{C}^+.$$

Consequently, by Equations (6) and (9), we have:

$$G_{\mu_s}(\xi) = tr_n(G_s(\xi.1)) = \frac{1}{n} \sum_{k=1}^n \langle G_s(\xi.1)e_k, e_k \rangle_{\ell_2^n} = \frac{1}{n} \sum_{k=1}^n G_{\mu_k}(\xi) \quad \xi \in \mathbb{C}^+,$$

and by the Equation (1) it follows that:

$$\mu_s = \frac{1}{n} \sum_{k=1}^n \mu_k,$$

proving the desired result. \square

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